Comparison of Moderate Energy Proton-Proton Models*

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(Received 29 October 1962; revised manuscript received 22 July 1963)

The predictions of six recently proposed two-nucleon models are compared to 647 pieces of proton-proton scattering data in the energy range 10–320 MeV. It is found that the value of the goodness-of-fit parameter, χ^2, ranges from four to nineteen times its expected value.

I. INTRODUCTION

DURING the past two years, at least six models[1–6] have been proposed for the proton-proton data up to 310 MeV. Calculations are reportedly in progress utilizing one or more of these models for hyperon-nucleon and nucleon-antinucleon scattering, electro-disintegration of deuterium, nuclear matter, the binding energy of tritium, etc. Thus, it may be of interest to have a comparison of the respective abilities of the models to fit a fairly comprehensive set of proton-proton data.

II. THE MODELS

The models differ in significant aspects, as shown in Table I. The Bryan[1] model has a hard core, with phenomenological central, tensor, and spin-orbit potentials added onto the one-pion-exchange-potential (OPEP). The Hamada[2] and Hamada-Johnston[6] (HJ) models have, in addition to the above forms, a phenomenological quadratic spin-orbit potential. The Yale[5] model is of the same general type as Hamada and HJ except that it excludes the spin-orbit potential for higher values of the total angular momentum J,[7] making comparison with other models difficult.[8] The Saylor-Bryan-Marshak[3] (SBM) model consists of boundary conditions plus the TMO[9] second and fourth order potentials (central and tensor only). The Feshbach-Lomon-Tubis[4] (FLT) model uses boundary conditions and a potential halfway between the TMO and BW[10] potentials (again, central and tensor only). The phase shifts predicted by Bryan, SBM, and FLT approach those of OPEP at high-orbital angular-momentum L and low energy. The other three models contain quadratic spin-orbit operators, proportional to L^2 in at least some states. The phase shifts for such states approach the corresponding OPEP values more slowly. Each of the three quadratic spin-orbit operators was defined differently.

The potentials are compared graphically in Figs. 1–5. As previously cautioned,[8] however, one must not compare the inner region of the Yale central, spin-orbit, or quadratic spin-orbit potentials with the corresponding parts of the other models.

III. DATA USED AND RESULTS

We have used 647 pieces of proton-proton scattering data in the energy range 10–320 MeV. This is all of the data in the references used by Breit *et al.*[11] in obtaining their YLAM and YRB1 phase-shift sets.

Cross-section angular distributions were treated relatively, the absolute normalization for each distribution being treated as a separate experimental parameter. Similarly, first target (polarizer) polarization was treated as a datum separate from the angular distribution of the relative polarization data. In practice, an absolute angular distribution was first predicted; then the predicted normalization was determined as that

TABLE I. Characteristics of the models. B.C. indicates a boundary condition at the indicated radius. Q_{12} is a quadratic spin-orbit operator.

Model	Core ($\hbar/M_\pi c$)	Singlet Q_{12}	$L\cdot S$	Triplet Q_{12}	L^2
Yale	0.35	x	$J\leqslant 2$		
Hamada-Johnston	0.34	x	x	x	x
FLT	0.50 B.C.				
SBM	0.53 B.C.				
Hamada	0.34	x	x	x	
Bryan	0.28, 0.38		x		

* Supported in part by the U. S. Atomic Energy Commission.
[1] R. A. Bryan, Nuovo Cimento **16**, 895 (1960).
[2] T. Hamada, Progr. Theoret. Phys. (Kyoto) 24, 1033 (1960).
[3] D. P. Saylor, R. A. Bryan, and R. E. Marshak, Phys. Rev. Letters **5**, 266 (1960). [The coupled boundary values stated in this reference were found to contain transcriptional errors. The corrected values, corresponding to the model and phase shifts actually given in the reference are: $f_{21}=0.28$, $f_{23}=12.5$, $f_2{}^t=-2.6$, $f_{43}=-2.0$, $f_{45}=2.2$, $f_4{}^t=(0.8)$.]
[4] H. Feshbach, E. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961).
[5] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).
[6] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).
[7] See footnote to Table I, Ref. 5.
[8] As an illustration of the difficulty encountered in attempting to graph the Yale potential, consider representing the cutoff function (up to a finite value of J) by a power series in J^2. For nonrelativistic Schrödinger theory, the potentials must be written in terms of the orbital angular momentum, L^2, rather than J^2. Thus, one must expand $(J^2)^n=(L^2+S^2+2L\cdot S)^n$ for each term in the cutoff polynomial, yielding additional (L^2-dependent) contributions to the central and quadratic spin-orbit potentials. The potential indicated as spin-orbit would be changed as well. [For a discussion of L dependence in potentials, see S. Okubo and R. E. Marshak, Ann. Phys. (N. Y.) **4**, 166 (1958).]
[9] M. Taketani, S. Machida, and S. Ohnuma, Prog. Theoret. Phys. (Kyoto) **6**, 638 (1951); **7**, 45 (1952).
[10] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1032 (1953).
[11] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960).

value which produced a least-squares (normalized) fit to the experimental angular distribution data.

Tables II and III display the goodness-of-fit parameter,[12] χ^2, which we have computed[13] *from the model parameters* for each of the six models. There is a statistical probability of approximately 0.001 of obtaining a $\chi^2 \geqslant 760$ for the 647 pieces of data used in this analysis. Smaller probabilities, corresponding to the higher values of total χ^2 in Table II, cannot be evaluated from this number of data.

The models fit the data in various ways. The Yale model does not fit $R(\theta)$ at 147 and 213 MeV, whereas Hamada fits it very well. All models are hard-pressed by accurate cross-section shapes, but the Yale model a little less so than the others. However, some predict cross sections too small at small angles, while others are too large there. The Bryan and HJ models give much too low a polarization normalization for the Harvard data; the other models less so. Finally, Hamada and HJ have considerably more difficulty with $E(\theta)$ at 213 MeV than do the other models.

The FLT attempt was, in part, concerned with the phenomenological evaluation of certain constants suggested by field theory. Setting those constants equal to zero or unity yields the earlier SBM model, except for differing numerical values for the purely phenomenological boundary conditions. Thus, the FLT model can be thought of as a later and more sophisticated[14] SBM model, the difference being merely in parameter values. Table II shows, however, that SBM is a considerably better fit to the data than is FLT. In order to understand this, we examined the phases "close to YLAM" which FLT tried to match. Table IV shows the χ^2 obtained for each of the models, using as "data" the true YLAM phases and the "close to YLAM" phases used

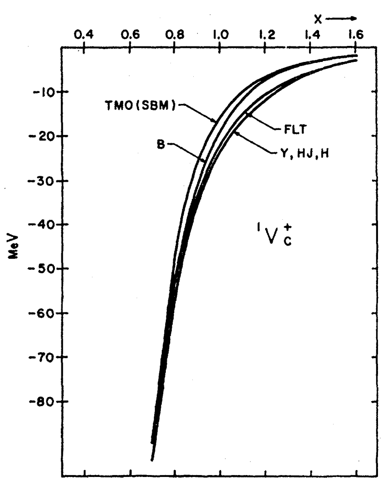

FIG. 1. Singlet, even parity, central potentials, versus $x = M_\pi cr/\hbar$.

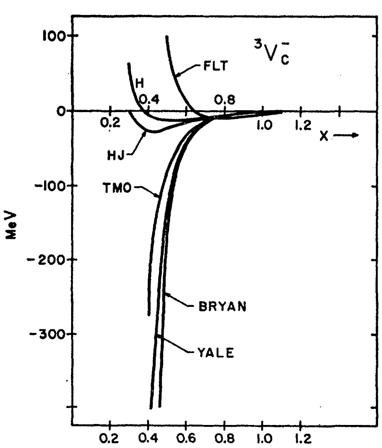

FIG. 2. Triplet, odd parity, central potentials.

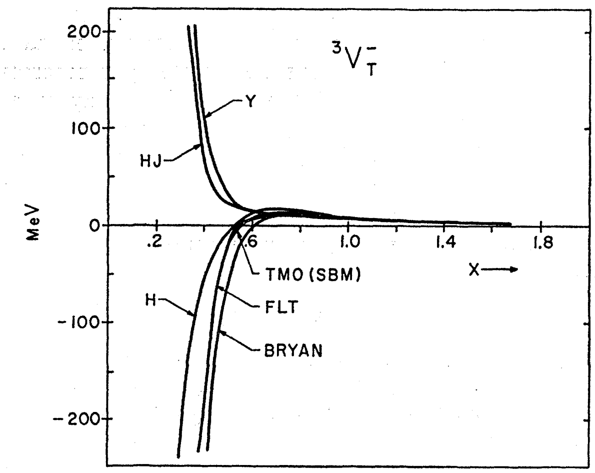

FIG. 3. Triplet, odd parity, tensor potentials (to be multiplied by $\langle S_{12}\rangle$).

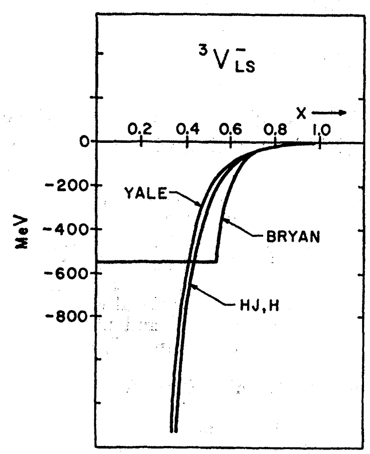

FIG. 4. Triplet, odd parity, spin-orbit potentials (to be multiplied by $\langle L\cdot S\rangle$).

[12] Called q in P. Cziffra and M. Moravcsik, University of California Radiation Laboratory Report, UCRL-8523 Rev., 1959 (unpublished).

[13] The phase shifts for each model were computed, from the potentials and boundary conditions, up to $L=5$ for energies less than 100 MeV and up to $L=7$ for energies greater than 100 MeV. A one-pion-exchange amplitude with the constants as used by the original authors was used to represent the higher angular momentum states.

[14] G. Breit, Rev. Mod. Phys. 34, 766 (1962). See especially p. 806–7.

by FLT. The FLT model does indeed fit the FLT-YLAM phases better than does the SBM model, but the latter is a better fit to the true YLAM phases. As a check, we computed the χ^2 fit to the experimental data from the YLAM phase shifts, and found that the true YLAM set gave a better fit than did the FLT-YLAM set.

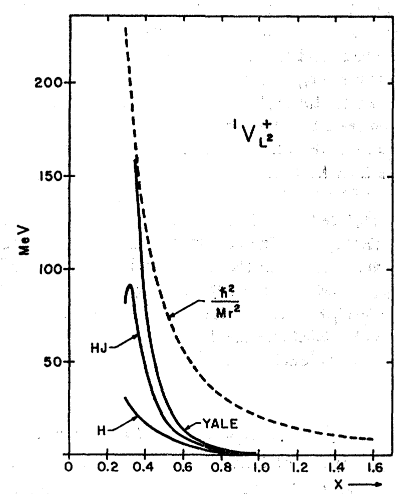

FIG. 5. Singlet, even parity, quadratic spin-orbit potentials (to be multiplied by $\langle L^2 \rangle$).

TABLE II. Model fits to the 647 pieces of 10–320-MeV proton-proton scattering data. The older BGT model[a] prediction is shown for comparison. Fits at individual energies are shown in Table IV. The number of degrees of freedom is the number of data minus the number of model parameters.[b]

Model		χ^2	No. of model parameters	χ^2/deg. of freedom
Yale	(1962)	2477	31	4.0
Hamada-Johnston	(1962)	3061	15	4.7
Feshback-Lomon-Tubis	(1961)	12 118	12	19.2
Saylor-Bryan-Marshak	(1960)	4454	13	7.0
Hamada	(1960)	3763	16	6.0
Bryan	(1960)	7900	14	12.5
Brueckner-Gammel-Thaler	(1958)	37 678	9	59.0

[a] K. A. Brueckner, J. A. Gammel, and R. M. Thaler, Phys. Rev. **109**, 1023 (1958).
[b] The model parameters listed in Table II are those which were adjusted to obtain a fit to the p-p data. One could define a "phenomenological figure of merit" for the models as (χ_{model}^2 times the number of model parameters)$^{-1}$. This would place the models in the order: HJ, Hamada and SBM, Yale, Bryan, and FLT.

IV. DISCUSSION

The "expected value" of χ^2, which corresponds to equal probabilities of obtaining a smaller or larger value of χ^2, is equal to the number of degrees of freedom. The latter, defined as the number of pieces of data used minus the number of free parameters in the model, is about 630 here. Thus, there is considerable statistical discrepancy between the model predictions and the data. A very small part of this could be removed by eliminating the small-angle data with its uncertainty in the treatment of electromagnetic effects. The major part of the discrepancy would remain.

TABLE III. Energy distributions of the χ^2 values of Table II.

Lab energy	No. Data	Yale	HJ	H	SBM	Bryan	FLT	BGT
9.69	28	41.0	20.2	20.0	121.7	283	117	842
14.16	18	1.6	1.5	1.5	2.3	2	6	22
18.2	9	18.6	16.1	15.5	73.3	27	57	131
19.8	25	25.9	21.1	22.4	54.8	18	43	334
25.63	24	38.3	41.6	42.0	201.3	234	430	3887
39.4	28	122.4	289.8	329.0	521.1	1050	1384	6410
44–57	6	10.8	12.4	15.4	201.8	17	14	184
66.0	24	36.3	197.6	232.5	255.6	1055	444	190
68.3	27	202.2	93.3	154.8	424.8	776	2090	5350
68.42	1	1.8	5.0	12.2	83.3	0	26	39
78	3	2.1	9.7	12.5	27.8	33	20	75
95	30	28.4	54.6	74.4	97.4	316	486	564
98	41	688.0	754.6	778.3	711.3	1045	1180	2143
102	8	7.2	57.5	92.1	77.5	344	216	205
107	8	17.8	23.1	41.8	42.7	181	159	290
118	34	75.7	45.2	119.2	77.7	163	1166	1831
120	4	2.8	1.1	1.0	7.0	1	13	18
127	8	17.0	33.7	66.6	55.7	279	227	354
133	8	22.6	22.4	22.9	19.6	17	23	32
134	1	5.3	1.3	0.0	15.0	1	26	62
137	8	7.4	10.2	27.4	23.1	103	155	320
140.5	6	31.7	10.2	6.3	17.3	45	6	199
142	81	391.6	609.3	754.4	586.6	755	1496	4801
143	7	10.4	22.6	28.7	5.9	4	10	27
147	63	129.2	163.6	205.8	185.4	236	1606	4876
164	1	3.4	1.4	0.2	5.7	0	9	41
170	15	38.3	68.0	56.2	60.3	48	62	221
174	14	38.0	90.9	67.8	69.7	26	56	246
210	15	18.8	19.8	25.1	19.5	48	18	46
213	27	245.5	102.9	303.3	172.3	464	159	1079
240	19	76.0	97.6	127.0	103.4	164	235	1424
250	5	2.7	4.5	2.5	4.7	6	6	34
259.5	8	35.5	101.4	47.0	22.3	53	48	808
276	7	9.5	10.1	14.6	22.2	14	16	31
310	27	63.6	40.6	34.3	66.5	79	50	553
315	7	3.3	4.9	5.4	14.1	7	20	18
316	3	6.0	0.7	2.6	3.1	6	39	11

TABLE IV. χ^2 fit of the boundary condition models to the YLAM phases.

Model	χ^2 fit to FLT-YLAM	χ^2 fit to true YLAM
FLT	31.5	203.0
SBM	45.6	125.0

It should be remembered, however, that if a two-nucleon model is to be used for calculations of other phenomena, one should examine the sensitivity of the predictions of such calculations to the two-nucleon model parameters. Only then is it possible to judge the accuracy with which the two-nucleon model should fit the two-nucleon data.

ACKNOWLEDGMENT

We would like to thank Professor H. P. Noyes for pointing out several mistakes in our original data table. The calculations reported here were carried out at the Computation Center of The Pennsylvania State University.